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On $\mathcal{I}$-lacunary Statistical Convergence of Order α for Sequences of Sets

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Abstract. In this paper, following a very recent and new approach of [1] and [2] we further generalize recently introduced summability methods in [11] and introduce new notions, namely, $\mathcal{I}$-statistical convergence of order α and $\mathcal{I}$-lacunary statistical convergence of order α, where $0 < \alpha \leq 1$ for sequences of sets. We mainly study their relationship and also make some observations about these classes and in the way try to give a proof of theorem which is not proved in [31]. The study leaves a lot of interesting open problems.

1. Introduction

The idea of statistical convergence was given by Zygmund [32]in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [29] and Fast [6] and later reintroduced by Schoenberg [28] independently as follows:

If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m, n)$ denotes the cardinality of the set $K \cap [m, n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1, n)}{n} \text{ and } \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1, n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \frac{K(1, n)}{n}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\}$ has natural density zero.

Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy [8], Kolk [10],Šalát [18], Mursaleen [15], Savas([20], [21], [23], [24]) where more references on this important summability method can be found . Nuray and Rhoades [17] extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems. Ulusu and Nuray [30] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relations with Wiijsman statistical convergence.

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The idea of statistical convergence was further extended to I-convergence in [12] using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [3–5, 13, 26, 27] where many important references can be found.

A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

In another direction, a new type of convergence called lacunary statistical convergence was introduced in [9] as follows:

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L) if for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [9] the relation between lacunary statistical convergence and statistical convergence was established among other things. More results on this convergence can be seen from [14, 19, 22, 24, 25].

Recently in [4], we used ideals to introduce the concepts of I-statistical convergence and I-lacunary statistical convergence which naturally extend the notions of the above mentioned convergence. Also, Kişi and Savaş defined I- lacunary statistical convergence of sequence of sets.

On the other hand in [2] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order α, $0 < \alpha \leq 1$ was introduced by replacing n by n^α in the denominator in the definition of statistical convergence.

In this paper we combine the approaches of [11] and [1, 2] and introduce new and further general summability methods, namely, $\mathcal{I}$-statistical convergence of order α and $\mathcal{I}$-lacunary statistical convergence of order α for sequence of sets.

In this context it should be mentioned that the concept of lacunary statistical convergence of order α (which happens to be a special case of $\mathcal{I}$-lacunary statistical convergence of order α) for sequence of sets has also not studied till now. We mainly investigate their relationship and also make some observations about these classes and most importantly the study leaves a lot of interesting open problems.

2. Main Results

The following definitions and notions will be needed in the sequel.

Definition 1. A family $I \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:
(a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(b) $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$,

Definition 2. A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be an filter of $\mathbb{N}$ if the following conditions hold:
(a) $\phi \notin F$,
(b) $A, B \in F$ implies $A \cap B \in F$,
(c) $A \in F,\ A \subset B$ implies $B \in F$,

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} :\ \exists\, A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 3. A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 4. ([12]) Let $I \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. Then

the sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\} \in \mathcal{I}$.

Recently, Kişi and Savaş defined $\mathcal{I}$- statistical convergence and $\mathcal{I}$- lacunary statistical convergence of sequence of sets as follows :

Definition 5. Let (X, ρ) be a metric space and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}-$statistical convergent to A or $S(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\,(S(\mathcal{I}_W))$.

Definition 6. Let (X, ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}-$lacunary statistical convergent to A or $S_\theta(\mathcal{I}_W)$-convergent to A if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\,(S_\theta(\mathcal{I}_W))$.

We now introduce our main definitions.

Definition 7. Let (X, ρ) be a metric space and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $I-$statistical convergent of order α to A or $S(\mathcal{I}_W)^\alpha$-convergent to A, where $0 < \alpha \leq 1$, if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A(S(\mathcal{I}_W)^\alpha)$.

The class of all $\mathcal{I}_W$-statistically convergent sequences of order α will be denoted by simply $S(\mathcal{I}_W)^\alpha$.

Remark 1. For $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbf{N} : A \text{ is a finite subset }\}$, $S(\mathcal{I}_W)^\alpha$-convergence coincides with Wijsman statistical convergence of order α which has not studied till now. For an arbitrary ideal I and for $\alpha = 1$ it coincides with Wijsman $\mathcal{I}$-statistical convergence,[11] . When $\mathcal{I} = \mathcal{I}_{fin}$ and $\alpha = 1$ it becomes only Wijsman statistical convergence, [17].

Definition 8. Let (X, ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-lacunary statistically convergent sequences of order α to A or $S_\theta(\mathcal{I}_W)^\alpha$-convergent to A, where $0 < \alpha \leq 1$, if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\,(S_\theta(\mathcal{I}_W)^\alpha)$.

The class of all Wijsman $\mathcal{I}$-lacunary statistically convergent sequences of order α will be denoted by $S_\theta(\mathcal{I}_W)^\alpha$.

Remark 2. For $\alpha = 1$ the definition coincides with Wijsman I-lacunary statistical convergence of sequence of sets [11]. Further it must be noted that Wijsman lacunary statistical convergence of order α has not been studied till now. Obviously Wijsman lacunary statistical convergence of order α is a special case of Wijsman $\mathcal{I}$-lacunary statistical convergence of order α when we take $\mathcal{I} = \mathcal{I}_{fin}$. So properties of Wijsman lacunary statistical convergence of order α can be easily obtained from our results with obvious modifications.

Theorem 1. Let $0 < \alpha \le \beta \le 1$. Then $S(\mathcal{I}_W)^\alpha \subset S(\mathcal{I}_W)^\beta$.

Proof: Let $0 < \alpha \le \beta \le 1$. Then

$$\frac{|\{k \le n : |d(x,A_k) - d(x,A)| \ge \epsilon\}|}{n^\beta} \le \frac{|\{k \le n : |d(x,A_k) - d(x,A)| \ge \epsilon\}|}{n^\alpha}$$

and so for any $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{|\{k \le n : |d(x,A_k) - d(x,A)| \ge \epsilon\}|}{n^\beta} \ge \delta\} \subset \{n \in \mathbb{N} : \frac{|\{k \le n : |d(x,A_k) - d(x,A)| \ge \epsilon\}|}{n^\alpha} \ge \delta\}.$$

Hence if the set on the right hand side belongs to the ideal $\mathcal{I}$ then obviously the set on the left hand side also belongs to $\mathcal{I}$. This shows that $S(\mathcal{I}_W)^\alpha \subset S(\mathcal{I}_W)^\beta$.

Corollary 1. If a sequence is Wijsman $\mathcal{I}$- statistically convergent of order α to A for some $0 < \alpha \le 1$ then it is Wijsman $\mathcal{I}$- statistically convergent i.e. $S(\mathcal{I}_W)^\alpha \subset S(\mathcal{I}_W)$.

Similarly we can show that

Theorem 2. Let $0 < \alpha \le \beta \le 1$. Then

(i) $S_\theta(\mathcal{I}_W)^\alpha \subset S_\theta(\mathcal{I}_W)^\beta$.

(ii) In particular $S_\theta(\mathcal{I}_W)^\alpha \subset S_\theta(\mathcal{I}_W)$.

Definition 9.

Let (X,ρ) be a metric space, θ be lacunary sequence and $I \subseteq 2^{\mathbb{N}}$ be a non-trivial ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is said to be Wijsman strongly $\mathcal{I}-$ lacunary convergent to A or $N_\theta\,(I_W)$-convergent of order α to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x,A_k) - d(x,A)| \ge \varepsilon \right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\,(N_\theta\,(I_W)^\alpha)$ and the class of such sequences will be denoted by simply $N_\theta(I_W)^\alpha$.

Theorem 3. Let (X,ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be non-empty closed subsets of X. Then $A_k \to A(N_\theta(\mathcal{I}_W)^\alpha)$ implies $A_k \to A(S_\theta(\mathcal{I}_W)^\alpha)$.

Proof: If $\varepsilon > 0$ and $A_k \to L(N_\theta(I_W)^\alpha)$, we can write, for each $x \in X$

$$\sum_{k \in I_r} |d(x,A_k) - d(x,A)| \ge \sum_{k \in I_r, |d(x,A_k) - d(x,A)| \ge \varepsilon} |d(x,A_k) - d(x,A)| \ge \varepsilon |\{k \in I_r : \ |d(x,A_k) - d(x,A)| \ge \varepsilon\}|$$

$$\text{and so} \qquad \frac{1}{\varepsilon . h_r^\alpha} \sum_{k \in I_r} |d(x,A_k) - d(x,A)| \ge \frac{1}{h_r^\alpha}\, |\{k \in I_r : \ |d(x,A_k) - d(x,A)| \ge \varepsilon\}|.$$

Then for each $x \in X$ and for any $\delta > 0$

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\} \subseteq \{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \varepsilon.\delta\} \in \mathcal{I}.$$

This completes the proof.

Remark 4. In Theorem 2 [31] it was further proved that
(ii) $\{A_k\} \in l_\infty$ and $A_k \to A(S_\theta(\mathcal{I}_W))$ implies $A_k \to A(N_\theta(\mathcal{I}_W))$,

(iii) $S_\theta(\mathcal{I}_W) \cap l_\infty = N_\theta(\mathcal{I}_W) \cap l_\infty$, where l_∞ is the set of all bounded real sequences.
However whether these results remain true for $0 < \alpha < 1$ is not clear and we leave them as open problems.

We will now investigate the relationship between Wijsman $\mathcal{I}$-statistical and Wijsman $\mathcal{I}$-lacunary statistical convergence of order α.

Theorem 4. Let (X, ρ) be a metric space, θ be lacunary sequence and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be non-empty closed subsets of X.Then $A_k \to A(S(\mathcal{I}_W)^\alpha)$ implies $A_k \to A(S_\theta(\mathcal{I}_W)^\alpha)$ if $\liminf_r q_r^\alpha > 1$.

Proof: Suppose first that $\liminf_r q_r^\alpha > 1$. Then there exists $\sigma > 0$ such that $q_r^\alpha \geq 1 + \sigma$ for sufficiently large r which implies that

$$\frac{h_r^\alpha}{k_r^\alpha} \geq \frac{\sigma}{1 + \sigma}.$$

Since $A_k \to A(S(I_W)^\alpha)$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{k_r^\alpha} |\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \frac{1}{k_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|$$

$$\geq \frac{\sigma}{1 + \sigma} . \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|.$$

Then for any $\delta > 0$, we get

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \delta\}$$

$$\subseteq \{r \in \mathbb{N} : \frac{1}{k_r^\alpha} |\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \geq \frac{\delta\sigma}{(1 + \sigma)}\} \in \mathcal{I}.$$

This completes the proof.

Remark 5. The converse of this result is true for $\alpha = 1$ (see Theorem 3.4 [31]). However for $\alpha < 1$ it is not clear and we leave it as an open problem.

We now present two theorems which specify the sufficient conditions for the converse relation of Theorem 4 to be true. In this context it should be mentioned that it was not prove in [31] for the case $\alpha = 1$. For the next two results we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F(I)$, $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \in F(\mathcal{I})$.

Theorem 5. Let (X, ρ) be a metric space, θ be lacunary sequence and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be non-empty closed subsets of X. Then $A_k \to A(S_\theta(\mathcal{I}_W))$ implies $A_k \to A(S(\mathcal{I}_W))$ if $\limsup_r q_r < \infty$.

Proof: If $\limsup_r q_r < \infty$ then without any loss of generality we can assume that there exists a $0 < B < \infty$ such that $q_r < B$ for all $r \geq 1$. Suppose that $A_k \to A(S_\theta(\mathcal{I}_W))$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \epsilon\}| < \delta\}$$

and

$$T = \{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \epsilon\}| < \delta_1\}.$$

It is obvious from our assumption that $C \in F(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$A_j = \frac{1}{h_j}|\{k \in I_j : |d(x, A_k) - d(x, A)| \geq \epsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\frac{1}{n}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \epsilon\}| \leq \frac{1}{k_{r-1}}|\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \epsilon\}|$$

$$= \frac{1}{k_{r-1}}|\{k \in I_1 : |d(x, A_k) - d(x, A)| \geq \epsilon\}| + \cdots + \frac{1}{k_{r-1}}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \epsilon\}|$$

$$= \frac{k_1}{k_{r-1}}\frac{1}{h_1}|\{k \in I_1 : |d(x, A_k) - d(x, A)| \geq \epsilon\}| + \frac{k_2 - k_1}{k_{r-1}}\frac{1}{h_2}|\{k \in I_2 : |d(x, A_k) - d(x, A)| \geq \epsilon\}| + \cdots +$$

$$+\frac{k_r - k_{r-1}}{k_{r-1}}\frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \epsilon\}|$$

$$= \frac{k_1}{k_{r-1}}A_1 + \frac{k_2 - k_1}{k_{r-1}}A_2 + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}A_r$$

$$\leq \sup_{j \in C} A_j . \frac{k_r}{k_{r-1}} < B\delta.$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem.

Theorem 6. Let (X, ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be non-empty closed subsets of X. Then $A_k \to A(S_\theta(\mathcal{I}_W)^\alpha)$ implies $A_k \to A(S(\mathcal{I}_W)^\alpha)$ if $\sup_r \sum_{i=0}^{r-1} \frac{h_{i+1}^\alpha}{(k_{r-1})^\alpha} = B(say) < \infty$.

Proof: Suppose that $A_k \to A(S_\theta(\mathcal{I}_W)^\alpha)$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \{r \in \mathbb{N} : \frac{1}{{h_r}^\alpha}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \epsilon\}| < \delta\}$$

and

$$T = \{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \epsilon\}| < \delta_1\}.$$

It is obvious from our assumption that $C \in F(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$A_j = \frac{1}{{h_j}^\alpha}|\{k \in I_j : |d(x, A_k) - d(x, A)| \geq \epsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\frac{1}{n^\alpha}|\{k \leq n : |d(x,A_k) - d(x,A)| \geq \epsilon\}| \leq \frac{1}{k_{r-1}^\alpha}|\{k \leq k_r : |d(x,A_k) - d(x,A)| \geq \epsilon\}|$$

$$= \frac{1}{k_{r-1}^\alpha}|\{k \in I_1 : |d(x,A_k) - d(x,A)| \geq \epsilon\}| + \cdots + \frac{1}{k_{r-1}^\alpha}|\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \epsilon\}|$$

$$= \frac{k_1^\alpha}{k_{r-1}^\alpha}\frac{1}{h_1^\alpha}|\{k \in I_1 : |d(x,A_k) - d(x,A)| \geq \epsilon\}| + \frac{(k_2-k_1)^\alpha}{k_{r-1}^\alpha}\frac{1}{h_2^\alpha}|\{k \in I_2 : |d(x,A_k) - d(x,A)| \geq \epsilon\}| + \cdots +$$

$$+\frac{(k_r-k_{r-1})^\alpha}{k_{r-1}^\alpha}\frac{1}{h_r^\alpha}|\{k \in I_r : |d(x,A_k) - d(x,A)| \geq \epsilon\}|$$

$$= \frac{k_1^\alpha}{k_{r-1}^\alpha}A_1 + \frac{(k_2-k_1)^\alpha}{k_{r-1}^\alpha}A_2 + \cdots + \frac{(k_r-k_{r-1})^\alpha}{k_{r-1}^\alpha}A_r$$

$$\leq \sup_{j\in C} A_j . \sup_r \sum_{i=0}^{r-1} \frac{(k_{i+1}-k_i)^\alpha}{k_{r-1}^\alpha} < B\delta.$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem.

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